

# ON RATE OF CONVERGENCE IN NON-CENTRAL LIMIT THEOREMS

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The main result of this paper is the rate of convergence to Hermite-type distributions in non-central limit theorems. To the best of our knowledge, this is the first result in the literature on rates of convergence of functionals of random fields to Hermite-type distributions with ranks greater than 2. The results were obtained under rather general assumptions on the spectral densities of random fields. These assumptions are even weaker than in the known convergence results for the case of Rosenblatt distributions. Additionally, Lévy concentration functions for Hermite-type distributions were investigated.

**1. Introduction.** This research will focus on the rate of convergence of local functionals of real-valued homogeneous random fields with long-range dependence. Non-linear integral functionals on bounded sets of  $\mathbb{R}^d$  are studied. These functionals are important statistical tools in various fields of application, for example, image analysis, cosmology, finance, and geology. It was shown in [10], [34] and [35] that these functionals can produce non-Gaussian limits and require normalizing coefficients different from those in central limit theorems.

Since many modern statistical models are now designed to deal with non-Gaussian data, non-central limit theory is gaining more and more popularity. Some novel results using different models and asymptotic distributions were obtained during the past few years, see [1], [6], [22], [30], [34] and references therein. Despite such development of the asymptotic theory, only a few of

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the studies obtained the rate of convergence, especially in the non-central case.

There are two popular approaches to investigate the rate of convergence in the literature: the direct probability approach [1], [17], and the Stein-Malliavin method introduced in [25].

As the name suggests, the Stein-Malliavin method combines Malliavin calculus and Stein's method. The main strength of this approach is that it does not use any restrictions on the moments of order higher than four (see, for example, [25]) and even three in some cases (see [23]). For a more detailed description of the method, the reader is referred to [25]. At this moment, the Stein-Malliavin approach is well developed for stochastic processes. However, many problems concerning non-central limit theorems for random fields remain unsolved. The full list of the already solved problems can be found in [37].

One of the first papers which obtained the rate of convergence in the central limit theorem using the Stein-Malliavin approach was [25]. The case of stochastic processes was considered. Further refinement of these results can be found in [26], where optimal Berry-Esseen bounds for the normal approximation of functionals of Gaussian fields are shown. However, it is known that numerous functionals do not converge to the Gaussian distribution. The conditions to obtain the Gaussian asymptotics can be found in so-called Breuer-Major theorems, see [2] and [11]. These results are based on the method of cumulants and diagram formulae. Using the Stein-Malliavin approach, [27] derived a version of a quantitative Breuer-Major theorem that contains a stronger version of the results in [2] and [11]. The rate of convergence for Wasserstein topology was found and an upper bound for the Kolmogorov distance was given as a relationship between the Kolmogorov and Wasserstein distances. In [16] the authors directly derived the upper-bound for the Kolmogorov distance in the same quantitative Breuer-Major theorem as in [27] and showed that this bound is better than the known bounds in the literature, since it converges to zero faster. The results described above are the most general results currently known concerning the rate of convergence in the central limit theorem using the Stein-Malliavin approach.

Related to [27] is the work [32] where, using the same arguments, the author found the rate of convergence for the central limit theorem of sojourn times of Gaussian fields. Similar results for the Kolmogorov distance were obtained in [16].

Concerning non-central limit theorems, only partial results have been found. It is known from [8],[11] and [34] that, depending on the value of the

Hurst parameter, functionals of fractional Brownian motion can converge either to the standard Gaussian distribution or a Hermite-type distribution. This idea was used in [6] and [7] to obtain the first rates of convergence in non-central limit theorems using the Stein-Malliavin method. Similar to the case of central limit theorems, these results were obtained for stochastic processes. In [7] fractional Brownian motion was considered, and rates of convergence for both Gaussian and Hermite-type asymptotic distributions were given. Furthermore all the results of [7] were refined in [6] for the case of the fractional Brownian sheet as an initial random element. It makes [7] the only known work that uses the Stein-Malliavin method to provide the rate of convergence of some local functionals of random fields with long-range dependence.

Separately stands [3]. This work followed a new approach based only on Stein's method without Malliavin calculus. The authors worked with Wasserstein-2 metrics and showed the rate of convergence of quadratic functionals of i.i.d. Gaussian variables. It is one of the convergence results which can't be obtained using the regular Stein-Malliavin method [3]. However, we are not aware of extensions of these results to the multi-dimensional and non-Gaussian cases.

The classical probability approach employs direct probability methods to find the rate of convergence. Its main advantage over the other methods is that it directly uses the correlation functions and spectral densities of the involved random fields. Therefore, asymptotic results can be explicitly obtained for wide classes of random fields using slowly varying functions. Using this approach, the first rate of convergence in the central limit theorem for Gaussian fields was obtained in [17]. In the following years, some other results were obtained, but all of them studied the convergence to the Gaussian distribution.

As for convergence to non-Gaussian distributions, the only known result using the classical probability approach is [1]. For functionals of Hermite rank-2 polynomials of long-range dependent Gaussian fields, it investigated the rate of convergence in the Kolmogorov metric of these functionals to the Rosenblatt-type distribution. In this paper, we generalize these results to some classes of Hermite-type distributions. It is worth mentioning that our present results are obtained under more natural and much weaker assumptions on the spectral densities than those in [1]. These quite general assumptions allow to consider various new asymptotic scenarios even for the Rosenblatt-type case in [1].

It's also worth mentioning that in the known Stein-Malliavin results, the rate of convergence was obtained only for a leading term or a fixed number

of chaoses in the Wiener chaos expansion. However, while other expansion terms in higher level Wiener chaoses do not change the asymptotic distribution, they can substantially contribute to the rate of convergence. The method proposed in this manuscript takes into account all terms in the Wiener chaos expansion to derive rates of convergence.

It is well known, see [8, 24, 33], that the probability distributions of Hermite-type random variables are absolutely continuous. In this paper we investigate some fine properties of these distributions required to derive rates of convergence. Specifically, we discuss the cases of bounded probability density functions of Hermite-type random variables. Using the method proposed in [28], we derive the anti-concentration inequality that can be applied to estimate the Lévy concentration function of Hermite-type random variables.

The article is organized as follows. In Section 2 we recall some basic definitions and formulae of the spectral theory of random fields. The main assumptions and auxiliary results are stated in Section 3. In Section 4 we discuss some fine properties of Hermite-type distributions. Section 5 provides the results concerning the rate of convergence. Discussions and conclusions are presented in Section 6.

**2. Notations.** In what follows  $|\cdot|$  and  $\|\cdot\|$  denote the Lebesgue measure and the Euclidean distance in  $\mathbb{R}^d$ , respectively. We use the symbols  $C$  and  $\delta$  to denote constants which are not important for our exposition. Moreover, the same symbol may be used for different constants appearing in the same proof.

We consider a measurable mean-square continuous zero-mean homogeneous isotropic real-valued random field  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , defined on a probability space  $(\Omega, \mathcal{F}, P)$ , with the covariance function

$$B(r) := \text{Cov}(\eta(x), \eta(y)) = \int_0^\infty Y_d(rz) d\Phi(z), \quad x, y \in \mathbb{R}^d,$$

where  $r := \|x - y\|$ ,  $\Phi(\cdot)$  is the isotropic spectral measure, the function  $Y_d(\cdot)$  is defined by

$$Y_d(z) := 2^{(d-2)/2} \Gamma\left(\frac{d}{2}\right) J_{(d-2)/2}(z) z^{(2-d)/2}, \quad z \geq 0,$$

$J_{(d-2)/2}(\cdot)$  being the Bessel function of the first kind of order  $(d-2)/2$ .

**DEFINITION 1.** The random field  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , as defined above is said to possess an absolutely continuous spectrum if there exists a function  $f(\cdot)$

such that

$$\Phi(z) = 2\pi^{d/2}\Gamma^{-1}(d/2) \int_0^z u^{d-1} f(u) du, \quad z \geq 0, \quad u^{d-1} f(u) \in L_1(\mathbb{R}_+).$$

The function  $f(\cdot)$  is called the isotropic spectral density function of the field  $\eta(x)$ . In this case, the field  $\eta(x)$  with an absolutely continuous spectrum has the isonormal spectral representation

$$\eta(x) = \int_{\mathbb{R}^d} e^{i(\lambda, x)} \sqrt{f(\|\lambda\|)} W(d\lambda),$$

where  $W(\cdot)$  is the complex Gaussian white noise random measure on  $\mathbb{R}^d$ .

Consider a Jordan-measurable bounded set  $\Delta \subset \mathbb{R}^d$  such that  $|\Delta| > 0$  and  $\Delta$  contains the origin in its interior. Let  $\Delta(r), r > 0$ , be the homothetic image of the set  $\Delta$ , with the centre of homothety at the origin and the coefficient  $r > 0$ , that is  $|\Delta(r)| = r^d |\Delta|$ .

Consider the uniform distribution on  $\Delta(r)$  with the probability density function (pdf)  $r^{-d} |\Delta|^{-1} \chi_{\Delta(r)}(x)$ ,  $x \in \mathbb{R}^d$ , where  $\chi_A(\cdot)$  is the indicator function of a set  $A$ .

**DEFINITION 2.** Let  $U$  and  $V$  be two random vectors which are independent and uniformly distributed inside the set  $\Delta(r)$ . We denote by  $\psi_{\Delta(r)}(z)$ ,  $z \geq 0$ , the pdf of the distance  $\|U - V\|$  between  $U$  and  $V$ .

Note that  $\psi_{\Delta(r)}(z) = 0$  if  $z > \text{diam}\{\Delta(r)\}$ . Using the above notations, one can obtain the representation

$$\begin{aligned} \int_{\Delta(r)} \int_{\Delta(r)} \mathcal{T}(\|x - y\|) dx dy &= |\Delta|^2 r^{2d} \mathbf{E} \mathcal{T}(\|U - V\|) \\ (2.1) \quad &= |\Delta|^2 r^{2d} \int_0^{\text{diam}\{\Delta(r)\}} \mathcal{T}(z) \psi_{\Delta(r)}(z) dz, \end{aligned}$$

where  $\mathcal{T}(\cdot)$  is an integrable Borel function.

**REMARK 1.** If  $\Delta(r)$  is the ball  $v(r) := \{x \in \mathbb{R}^d : \|x\| < r\}$ , then

$$\psi_{v(r)}(z) = d r^{-d} z^{d-1} I_{1-(z/2r)^2} \left( \frac{d+1}{2}, \frac{1}{2} \right), \quad 0 \leq z \leq 2r,$$

where

$$I_\mu(p, q) := \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^\mu u^{p-1} (1-u)^{q-1} du, \quad \mu \in (0, 1], \quad p > 0, \quad q > 0,$$

is the incomplete beta function, see [15].

REMARK 2. Let  $H_k(u)$ ,  $k \geq 0$ ,  $u \in \mathbb{R}$ , be the Hermite polynomials, see [30]. If  $(\xi_1, \dots, \xi_{2p})$  is a  $2p$ -dimensional zero-mean Gaussian vector with

$$\mathbf{E}\xi_j\xi_k = \begin{cases} 1, & \text{if } k = j, \\ r_j, & \text{if } k = j + p \text{ and } 1 \leq j \leq p, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\mathbf{E} \prod_{j=1}^p H_{k_j}(\xi_j) H_{m_j}(\xi_{j+p}) = \prod_{j=1}^p \delta_{k_j}^{m_j} k_j! r_j^{k_j}.$$

The Hermite polynomials form a complete orthogonal system in the Hilbert space

$$L_2(\mathbb{R}, \phi(w) dw) = \left\{ G : \int_{\mathbb{R}} G^2(w) \phi(w) dw < \infty \right\}, \quad \phi(w) := \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}.$$

An arbitrary function  $G(w) \in L_2(\mathbb{R}, \phi(w) dw)$  admits the mean-square convergent expansion

$$(2.2) \quad G(w) = \sum_{j=0}^{\infty} \frac{C_j H_j(w)}{j!}, \quad C_j := \int_{\mathbb{R}} G(w) H_j(w) \phi(w) dw.$$

By Parseval's identity

$$(2.3) \quad \sum_{j=0}^{\infty} \frac{C_j^2}{j!} = \int_{\mathbb{R}} G^2(w) \phi(w) dw.$$

DEFINITION 3. [34] Let  $G(w) \in L_2(\mathbb{R}, \phi(w) dw)$  and assume there exists an integer  $\kappa \in \mathbb{N}$  such that  $C_j = 0$ , for all  $0 \leq j \leq \kappa - 1$ , but  $C_{\kappa} \neq 0$ . Then  $\kappa$  is called the Hermite rank of  $G(\cdot)$  and is denoted by  $H\text{rank } G$ .

DEFINITION 4. [4] A measurable function  $L : (0, \infty) \rightarrow (0, \infty)$  is said to be slowly varying at infinity if for all  $t > 0$ ,

$$\lim_{r \rightarrow \infty} \frac{L(rt)}{L(r)} = 1.$$

By the representation theorem [4, Theorem 1.3.1], there exists  $C > 0$  such that for all  $r \geq C$  the function  $L(\cdot)$  can be written in the form

$$L(r) = \exp \left( \zeta_1(r) + \int_C^r \frac{\zeta_2(u)}{u} du \right),$$

where  $\zeta_1(\cdot)$  and  $\zeta_2(\cdot)$  are such measurable and bounded functions that  $\zeta_2(r) \rightarrow 0$  and  $\zeta_1(r) \rightarrow C_0$  ( $|C_0| < \infty$ ), when  $r \rightarrow \infty$ .

If  $L(\cdot)$  varies slowly, then  $r^a L(r) \rightarrow \infty$ ,  $r^{-a} L(r) \rightarrow 0$  for an arbitrary  $a > 0$  when  $r \rightarrow \infty$ , see Proposition 1.3.6 [4].

DEFINITION 5. [4] A measurable function  $g : (0, \infty) \rightarrow (0, \infty)$  is said to be regularly varying at infinity, denoted  $g(\cdot) \in R_\tau$ , if there exists  $\tau$  such that, for all  $t > 0$ , it holds that

$$\lim_{r \rightarrow \infty} \frac{g(rt)}{g(r)} = t^\tau.$$

DEFINITION 6. [4] Let  $g : (0, \infty) \rightarrow (0, \infty)$  be a measurable function and  $g(x) \rightarrow 0$  as  $x \rightarrow 0$ . Then a slowly varying function  $L(\cdot)$  is said to be slowly varying with remainder of type 2, or that it belongs to the class SR2, if

$$\forall x > 1 : \quad \frac{L(rx)}{L(r)} - 1 \sim k(x)g(r), \quad r \rightarrow \infty,$$

for some function  $k(\cdot)$ .

If there exists  $x$  such that  $k(x) \neq 0$  and  $k(x\mu) \neq k(\mu)$  for all  $\mu$ , then  $g(\cdot) \in R_\tau$  for some  $\tau \leq 0$  and  $k(x) = ch_\tau(x)$ , where

$$(2.4) \quad h_\tau(x) = \begin{cases} \ln(x), & \text{if } \tau = 0, \\ \frac{x^\tau - 1}{\tau}, & \text{if } \tau \neq 0. \end{cases}$$

**3. Assumptions and auxiliary results.** In this section, we list the main assumptions and some auxiliary results from [20] which will be used to obtain the rate of convergence in non-central limit theorems.

ASSUMPTION 1. Let  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , be a homogeneous isotropic Gaussian random field with  $\mathbf{E}\eta(x) = 0$  and a covariance function  $B(x)$  such that

$$B(0) = 1, \quad B(x) = \mathbf{E}\eta(0)\eta(x) = \|x\|^{-\alpha} L(\|x\|),$$

where  $L(\|\cdot\|)$  is a function slowly varying at infinity.

In this paper we restrict our consideration to  $\alpha \in (0, d/\kappa)$ , where  $\kappa$  is the Hermite rank in Definition 3. For such  $\alpha$  the covariance function  $B(x)$  satisfying Assumption 1 is not integrable, which corresponds to the case of long-range dependence.

Let us denote

$$K_r := \int_{\Delta(r)} G(\eta(x)) \, dx \quad \text{and} \quad K_{r,\kappa} := \frac{C_\kappa}{\kappa!} \int_{\Delta(r)} H_\kappa(\eta(x)) \, dx,$$

where  $C_\kappa$  is defined by (2.2).

**THEOREM 1.** [20] *Suppose that  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , satisfies Assumption 1 and  $H\text{rank } G = \kappa \in \mathbb{N}$ . If at least one of the following random variables*

$$\frac{K_r}{\sqrt{\text{Var } K_r}}, \quad \frac{K_r}{\sqrt{\text{Var } K_{r,\kappa}}} \quad \text{and} \quad \frac{K_{r,\kappa}}{\sqrt{\text{Var } K_{r,\kappa}}},$$

*has a limit distribution, then the limit distributions of the other random variables also exist and they coincide when  $r \rightarrow \infty$ .*

**ASSUMPTION 2.** *The random field  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , has the spectral density*

$$f(\|\lambda\|) = c_2(d, \alpha) \|\lambda\|^{\alpha-d} L\left(\frac{1}{\|\lambda\|}\right),$$

where

$$c_2(d, \alpha) := \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2} \Gamma\left(\frac{\alpha}{2}\right)},$$

*and  $L(\|\cdot\|)$  is a locally bounded function which is slowly varying at infinity and satisfies for sufficiently large  $r$  the condition*

$$(3.1) \quad \left| 1 - \frac{L(tr)}{L(r)} \right| \leq C g(r) h_\tau(t), \quad t \geq 1,$$

*where  $g(\cdot) \in R_\tau$ ,  $\tau \leq 0$ , such that  $g(x) \rightarrow 0$ ,  $x \rightarrow \infty$ , and  $h_\tau(t)$  is defined by (2.4).*

**REMARK 3.** In applied statistical analysis of long-range dependent models researchers often assume an equivalence of Assumptions 1 and 2. However, this claim is not true in general, see [12, 19]. This is the main reason of using both assumptions to formulate the most general result in Theorem 5. However, in various specific cases just one of the assumptions may be sufficient. For example, if  $f(\cdot)$  is decreasing in a neighbourhood of zero and continuous for all  $\lambda \neq 0$ , then by Tauberian Theorem 4 [19] both assumptions are simultaneously satisfied. A detailed discussion of relations between Assumption 1 and 2 and various examples can be found in [19, 29]. Some important models used in spatial data analysis and geostatistics that



simultaneously satisfy Assumptions 1 and 2 are Cauchy and Linnik's fields, see [1]. Their covariance functions are of the form  $B(x) = (1 + \|x\|^\sigma)^{-\theta}$ ,  $\sigma \in (0, 2]$ ,  $\theta > 0$ . Exact expressions for their spectral densities in the form required by Assumption 2 are provided in Section 5 [1].

The remarks below clarify condition (3.1) and compare it with the assumptions used in [1].

REMARK 4. This assumption implies weaker restrictions on the spectral density than the ones used in [1]. Slowly varying functions in Assumption 2 can tend to infinity or zero. This is an improvement over [1] where slowly varying functions were assumed to converge to a constant. For example, a function that satisfies this assumption, but would not fit that of [1], is  $\ln(\cdot)$ .

REMARK 5. If we consider the equivalence in Definition 6 in the uniform sense, then all the functions in the class SR2 satisfy condition (3.1). If we consider this equivalence in the non-uniform sense, then there are functions from SR2 that do not satisfy (3.1). An example of such functions is  $\ln^2(\cdot)$ .

REMARK 6. By Corollary 3.12.3 [4] for  $\tau \neq 0$  the slowly varying function  $L(\cdot)$  in Assumption 2 can be represented as

$$L(x) = C \left( 1 + c\tau^{-1}g(x) + o(g(x)) \right).$$

As we can see  $L(\cdot)$  converges to some constant as  $x$  goes to infinity. This makes the case  $\tau = 0$  particularly interesting as this is the only case when a slowly varying function with remainder can tend to infinity or zero.

LEMMA 1. *If  $L$  satisfies (3.1), then for any  $k \in \mathbb{N}$ ,  $\delta > 0$ , and sufficiently large  $r$*

$$\left| 1 - \frac{L^{k/2}(tr)}{L^{k/2}(r)} \right| \leq C g(r) h_\tau(t) t^\delta, \quad t \geq 1.$$

PROOF. Applying the mean value theorem to the function  $f(u) = u^n$ ,  $n \in \mathbb{R}$ , on  $\mathbf{A} = [\min(1, u), \max(1, u)]$  we obtain the inequality

$$1 - x^n = n\theta^{n-1}(1 - x) \leq n(1 - x) \max(1, x^{n-1}), \quad \theta \in \mathbf{A}.$$

Now, using this inequality for  $x = \frac{L(tr)}{L(r)}$  and  $n = k/2$  we get

$$(3.2) \quad \left| 1 - \frac{L^{k/2}(tr)}{L^{k/2}(r)} \right| \leq n \left| 1 - \frac{L(tr)}{L(r)} \right| \max \left( 1, \left( \frac{L(tr)}{L(r)} \right)^{\frac{k}{2}-1} \right).$$

By Theorem 1.5.6 [4] we know there exists  $c > 0$  such that for any  $\delta_1 > 0$

$$\frac{L(tr)}{L(r)} \leq C \cdot t^{\delta_1}, \quad t \geq 1.$$

Applying this result and condition (3.1) to (3.2) we get

$$\left| 1 - \frac{L^{k/2}(tr)}{L^{k/2}(r)} \right| \leq C g(r) h_\tau(t) \max \left( 1, t^{\delta_1 \left( \frac{k}{2} - 1 \right)} \right) \leq C g(r) h_\tau(t) t^\delta, \quad t \geq 1.$$

□

Let us denote the Fourier transform of the indicator function of the set  $\Delta$  by

$$K_\Delta(x) := \int_\Delta e^{i(x,u)} du, \quad x \in \mathbb{R}^d.$$

LEMMA 2. [20] *If  $t_1, \dots, t_\kappa$ ,  $\kappa \geq 1$ , are positive constants such that it holds  $\sum_{i=1}^\kappa t_i < d$ , then*

$$\int_{\mathbb{R}^{d\kappa}} |K_\Delta(\lambda_1 + \dots + \lambda_\kappa)|^2 \frac{d\lambda_1 \dots d\lambda_\kappa}{\|\lambda_1\|^{d-t_1} \dots \|\lambda_\kappa\|^{d-t_\kappa}} < \infty.$$

THEOREM 2. [20] *Let  $\eta(x)$ ,  $x \in \mathbb{R}^d$ , be a homogeneous isotropic Gaussian random field with  $\mathbf{E}\eta(x) = 0$ . If Assumptions 1 and 2 hold, then for  $r \rightarrow \infty$  the finite-dimensional distributions of*

$$X_{r,\kappa} := r^{(\kappa\alpha)/2-d} L^{-\kappa/2}(r) \int_{\Delta(r)} H_\kappa(\eta(x)) dx$$

*converge weakly to the finite-dimensional distributions of*

$$(3.3) \quad X_\kappa(\Delta) := c_2^{\kappa/2}(d, \alpha) \int'_{\mathbb{R}^{d\kappa}} K_\Delta(\lambda_1 + \dots + \lambda_\kappa) \times \frac{W(d\lambda_1) \dots W(d\lambda_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \dots \|\lambda_\kappa\|^{(d-\alpha)/2}},$$

*where  $\int'_{\mathbb{R}^{d\kappa}}$  denotes the multiple Wiener-Itô integral.*

REMARK 7. If  $\kappa = 1$  the limit  $X_\kappa(\Delta)$  is Gaussian. However, for the case  $\kappa > 1$  distributional properties of  $X_\kappa(\Delta)$  are almost unknown. It was shown that the integrals in (3.3) possess absolutely continuous densities, see [8, 33].

The article [1] proved that these densities are bounded if  $\kappa = 2$ . Also, for the Rosenblatt distribution, i.e.  $\kappa = 2$  and a rectangular  $\Delta$ , the density and cumulative distribution functions of  $X_\kappa(\Delta)$  were studied in [36]. An approach to investigate the boundedness of densities of multiple Wiener-Itô integrals was suggested in [8]. However, it is difficult to apply this approach to the case  $\kappa > 2$  as it requires a classification of the peculiarities of general  $n$ th degree forms.

**DEFINITION 7.** Let  $Y_1$  and  $Y_2$  be arbitrary random variables. The uniform (Kolmogorov) metric for the distributions of  $Y_1$  and  $Y_2$  is defined by the formula

$$\rho(Y_1, Y_2) = \sup_{z \in \mathbb{R}} |P(Y_1 \leq z) - P(Y_2 \leq z)|.$$

The following result follows from Lemma 1.8 [31].

**LEMMA 3.** *If  $X, Y$  and  $Z$  are arbitrary random variables, then for any  $\varepsilon > 0$ :*

$$\rho(X + Y, Z) \leq \rho(X, Z) + \rho(Z + \varepsilon, Z) + P(|Y| \geq \varepsilon).$$

**4. Lévy concentration functions for  $X_k(\Delta)$ .** In this section, we will investigate some fine properties of probability distributions of Hermite-type random variables. These results will be used to derive upper bounds of  $\rho(X_\kappa(\Delta) + \varepsilon, X_\kappa(\Delta))$  in the next section. The following function from Section 1.5 [31] will be used in this section.

**DEFINITION 8.** The Lévy concentration function of a random variable  $X$  is defined by

$$Q(X, \varepsilon) := \sup_{z \in \mathbb{R}} P(z < X \leq z + \varepsilon), \quad \varepsilon \geq 0.$$

We will discuss three important cases, and show how to estimate the Lévy concentration function in each of them.

If  $X_k(\Delta)$  has a bounded probability density function  $p_{X_\kappa(\Delta)}(\cdot)$ , then it holds

$$(4.1) \quad Q(X_\kappa(\Delta), \varepsilon) \leq \varepsilon \sup_{z \in \mathbb{R}} p_{X_\kappa(\Delta)}(z) \leq \varepsilon C.$$

This inequality is probably the sharpest known estimator of the Lévy concentration function of  $X_k(\Delta)$ . It is discussed in cases 1 and 2.

**Case 1.** If the Hermite rank of  $G(\cdot)$  is equal to  $\kappa = 2$  we are dealing with the so-called Rosenblatt-type random variable. It is known that the probability density function of this variable is bounded, consult [1, 8, 9, 18, 21] for proofs by different methods. Thus, one can use estimate (4.1).

**Case 2.** Some interesting results about boundedness of probability density functions of Hermite-type random variables were obtained in [14] by Malliavin calculus. To present these results we provide some definitions from Malliavin calculus.

Let  $X = \{X(h), h \in L^2(\mathbb{R}^d)\}$  be an isonormal Gaussian process defined on a complete probability space  $(\Omega, \mathcal{F}, P)$ . Let  $\mathbf{S}$  denote the class of smooth random variables of the form  $F = f(X(h_1), \dots, X(h_n))$ ,  $n \in \mathbb{N}$ , where  $h_1, \dots, h_n$  are in  $L^2(\mathbb{R}^d)$ , and  $f$  is a function, such that  $f$  itself and all its partial derivatives have at most polynomial growth.

The Malliavin derivative  $DF$  of  $F = f(X(h_1), \dots, X(h_n))$  is the  $L^2(\mathbb{R}^d)$  valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(h_1), \dots, X(h_n)) h_i.$$

The derivative operator  $D$  is a closable operator on  $L_2(\Omega)$  taking values in  $L_2(\Omega; L^2(\mathbb{R}^d))$ . By iteration one can define higher order derivatives  $D^k F \in L_2(\Omega; L^2(\mathbb{R}^d)^{\odot k})$ , where  $\odot$  denotes the symmetric tensor product. For any integer  $k \geq 0$  and any  $p \geq 1$  we denote by  $\mathbb{D}^{k,p}$  the closure of  $\mathbf{S}$  with respect to the norm  $\|\cdot\|_{k,p}$  given by

$$\|F\|_{k,p}^p = \sum_{i=0}^k \mathbf{E} \left( \|D^i F\|_{L^2(\mathbb{R}^d)^{\otimes i}}^p \right).$$

Let's denote by  $\delta$  the adjoint operator of  $D$  from a domain in  $L_2(\Omega; L^2(\mathbb{R}^d))$  to  $L_2(\Omega)$ . An element  $u \in L_2(\Omega; L^2(\mathbb{R}^d))$  belongs to the domain of  $\delta$  if and only if for any  $F \in \mathbb{D}^{1,2}$  it holds

$$\mathbf{E}[\langle DF, u \rangle] \leq c_u \sqrt{\mathbf{E}[F^2]},$$

where  $c_u$  is a constant depending only on  $u$ .

The following theorem gives sufficient conditions to guarantee boundedness of Hermite-type densities.

**THEOREM 3.** [14] *Let  $F \in \mathbb{D}^{2,s}$  such that  $\mathbf{E}[|F|^{2q}] < \infty$  and*

$$(4.2) \quad \mathbf{E} \left[ \|DF\|_{L^2(\mathbb{R}^d)}^{-2r} \right] < \infty,$$

where  $q, r, s > 1$  satisfying  $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1$ .

Denote  $w = \|DF\|_{L^2(\mathbb{R}^d)}^2$  and  $u = w^{-1}DF$ . Then  $u \in \mathbb{D}^{1,q'}$  with  $q' = \frac{q}{q-1}$  and  $F$  has a density given by  $p_F(x) = \mathbf{E}[\mathbf{1}_{F>x}\delta(u)]$ . Furthermore,  $p_F(x)$  is bounded and  $p_F(x) \leq C_q \|w^{-1}\|_r \|F\|_{2,s} \min(1, |x|^{-2} \|F\|_{2q}^2)$ , for any  $x \in \mathbb{R}$ , where  $C_q$  is a constant depending only on  $q$ .

Note, that the Hermite-type random variable  $X_\kappa(\Delta)$  does belong to the space  $\mathbb{D}^{2,s}$ ,  $s > 1$ , and  $\mathbf{E}[|X_\kappa(\Delta)|^{2q}] < \infty$  by the hypercontractivity property, see (2.11) in [14]. Thus, if the condition (4.2) holds, one can use (4.1).

**Case 3.** When there is no information about boundedness of the probability density function, anti-concentration inequalities can be used to obtain estimates of the Lévy concentration function.

Let us denote by  $I_\kappa(\cdot)$  a multiple Wiener-Itô stochastic integral of order  $d\kappa$ , i.e.  $I_\kappa(f) = \int_{\mathbb{R}^{d\kappa}} f(\lambda_1, \dots, \lambda_\kappa) W(d\lambda_1) \dots W(d\lambda_\kappa)$ , where  $f(\cdot) \in L_2^s(\mathbb{R}^{d\kappa})$ . Here  $L_2^s(\mathbb{R}^{d\kappa})$  denotes the space of symmetrical functions in  $L_2(\mathbb{R}^{d\kappa})$ . Note, that any  $F \in L_2(\Omega)$  can be represented as  $F = \mathbf{E}(F) + \sum_{q=1}^{\infty} I_q(f_q)$ , where the functions  $f_q$  are determined by  $F$ . The multiple Wiener-Itô integral  $I_q(f_q)$  coincides with the orthogonal projection of  $F$  on the  $q$ -th Wiener chaos associated with  $X$ .

The following lemma uses the approach suggested in [28].

LEMMA 4. For any  $\kappa \in \mathbb{N}$ ,  $t \in \mathbb{R}$ , and  $\hat{\varepsilon} > 0$  it holds

$$\mathbf{P}(|X_\kappa(\Delta) - t| \leq \hat{\varepsilon}) \leq \frac{c_\kappa \hat{\varepsilon}^{1/\kappa}}{\left(C \|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}^2 + t^2\right)^{1/\kappa}},$$

where  $\hat{K}_\Delta(x_1, \dots, x_\kappa) := \frac{K_\Delta(x_1 + \dots + x_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \dots \|\lambda_\kappa\|^{(d-\alpha)/2}}$  and  $c_\kappa$  is a constant that depends on  $\kappa$ .

PROOF. Let  $\{e_i\}_{i \in \mathbb{N}}$  be an orthogonal basis of  $L_2(\mathbb{R}^d)$ . Then,  $\hat{K}_\Delta \in L_2(\mathbb{R}^{d\kappa})$  can be represented as

$$\hat{K}_\Delta = \sum_{(i_1, \dots, i_\kappa) \in \mathbb{N}^\kappa} c_{i_1, \dots, i_\kappa} e_{i_1} \otimes \dots \otimes e_{i_\kappa}.$$

For each  $n \in \mathbb{N}$ , set

$$\hat{K}_\Delta^n = \sum_{(i_1, \dots, i_\kappa) \in \{1, \dots, n\}^\kappa} c_{i_1, \dots, i_\kappa} e_{i_1} \otimes \dots \otimes e_{i_\kappa}.$$

Note, that both  $\hat{K}_\Delta$  and  $\hat{K}_\Delta^n$  belong to the space  $L_2^s(\mathbb{R}^{d\kappa})$ .

By (3.3) it follows that  $X_\kappa(\Delta) = c_2^{\kappa/2}(d, \alpha)I_\kappa(\hat{K}_\Delta)$ . Let us denote  $X_\kappa^n(\Delta) := c_2^{\kappa/2}(d, \alpha)I_\kappa(\hat{K}_\Delta^n)$ .

As  $n \rightarrow \infty$ ,  $\hat{K}_\Delta^n \rightarrow \hat{K}_\Delta$  in  $L_2(\mathbb{R}^{d\kappa})$ . Thus,  $X_\kappa^n(\Delta) \rightarrow X_\kappa(\Delta)$  in  $L_2(\Omega, \mathcal{F}, P)$ . Hence, there exists a strictly increasing sequence  $n_j$  for which  $X_\kappa^{n_j}(\Delta) \rightarrow X_\kappa(\Delta)$  almost surely as  $j \rightarrow \infty$ .

It also follows that

$$\begin{aligned} X_\kappa^n(\Delta) &= c_2^{\kappa/2}(d, \alpha)I_\kappa \left( \sum_{(i_1, \dots, i_\kappa) \in \{1, \dots, n\}^\kappa} c_{i_1, \dots, i_\kappa} e_{i_1} \otimes \dots \otimes e_{i_\kappa} \right) \\ &= c_2^{\kappa/2}(d, \alpha) \sum_{m=1}^{\kappa} \sum_{\substack{1 \leq i'_1 < \dots < i'_m \leq n \\ \kappa_1 + \dots + \kappa_m = \kappa}}^n c_{i'_1, \dots, i'_m}^{\kappa_1, \dots, \kappa_m} I_\kappa(e_{i'_1}^{\otimes \kappa_1} \otimes \dots \otimes e_{i'_m}^{\otimes \kappa_m}), \end{aligned}$$

where  $\kappa_i \in \mathbb{N}$ ,  $i = 1, \dots, m$ ,  $c_{i'_1, \dots, i'_m}^{\kappa_1, \dots, \kappa_m} = \sum_{(i_1, \dots, i_\kappa) \in A_{i'_1, \dots, i'_m}^{\kappa_1, \dots, \kappa_m}} c_{i_1, \dots, i_\kappa}$ , and

$A_{i'_1, \dots, i'_m}^{\kappa_1, \dots, \kappa_m} := \{(i_1, \dots, i_\kappa) \in \{1, \dots, n\}^\kappa : \kappa_1 \text{ indices } i_l = i'_1, \dots, \kappa_m \text{ indices } i_l = i'_m, l = 1, \dots, \kappa\}$ .

By the Itô formula [15]:

$$I_{\kappa_1 + \dots + \kappa_m}(e_{i'_1}^{\otimes \kappa_1} \otimes \dots \otimes e_{i'_m}^{\otimes \kappa_m}) = \prod_{j=1}^m H_{\kappa_j} \left( \int_{\mathbb{R}^d} e_j(\lambda) W(d\lambda) \right) = \prod_{j=1}^m H_{\kappa_j}(\xi_j),$$

where  $\xi_j \sim \mathcal{N}(0, 1)$ .

Thus,  $X_\kappa^n(\Delta)$  can be represented as  $X_\kappa^n(\Delta) = U_{n, \kappa}(\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n)$ , where  $U_{n, \kappa}(\cdot)$  is a polynomial of the degree at most  $\kappa$ . Furthermore,  $X_\kappa^n(\Delta) - t$  is also a polynomial of the degree at most  $\kappa$ .

Now, applying Carbery-Wright inequality, see Theorem 2.5 [28], one obtains that there exists a constant  $\hat{c}_\kappa$  such that for any  $n \in \mathbb{N}$  and  $\hat{\varepsilon} > 0$

$$\mathbb{P} \left( |X_\kappa^n(\Delta) - t| \leq \hat{\varepsilon} \left( \mathbf{E}(X_\kappa^n(\Delta) - t)^2 \right)^{\frac{1}{2}} \right) \leq \hat{c}_\kappa \hat{\varepsilon}^{1/\kappa}.$$

Analogously to [28], using Fatou's lemma we get

$$\mathbb{P} \left( |X_\kappa(\Delta) - t| \leq \hat{\varepsilon} \left( \mathbf{E}(X_\kappa(\Delta) - t)^2 \right)^{\frac{1}{2}} \right) \leq \hat{c}_\kappa 2^{1/\kappa} \hat{\varepsilon}^{1/\kappa} = c_\kappa \hat{\varepsilon}^{1/\kappa}.$$

It is known, see (1.3) and (1.5) in [13], that  $\mathbf{E}X_\kappa(\Delta) = 0$  and  $\mathbf{E}(X_\kappa(\Delta))^2 = C\|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}^2$ . Thus, the above inequality can be rewritten as

$$\mathbf{P}(|X_\kappa(\Delta) - t| \leq \hat{\varepsilon}) \leq \frac{c_\kappa \hat{\varepsilon}^{1/\kappa}}{\left(\mathbf{E}(X_\kappa(\Delta) - t)^2\right)^{\frac{1}{2\kappa}}} = \frac{c_\kappa \hat{\varepsilon}^{1/\kappa}}{\left(C\|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}^2 + t^2\right)^{1/\kappa}}.$$

□

The following theorem combines all three cases above and provides an upper-bound estimator of the Lévy concentration function.

**THEOREM 4.** *For any  $\kappa \in \mathbb{N}$  and an arbitrary positive  $\varepsilon$  it holds*

$$Q(X_\kappa(\Delta), \varepsilon) \leq C\varepsilon^a,$$

where the constant  $a$  depends on the cases discussed above.

**PROOF.** For cases 1 and 2 it is an immediate corollary of (4.1) and the boundedness of  $p_{X_\kappa(\Delta)}(\cdot)$ .

For case 3, applying Lemma 4 with  $t = z + \frac{\varepsilon}{2}$  and  $\hat{\varepsilon} = \frac{\varepsilon}{2}$  we get

$$\begin{aligned} Q(X_\kappa(\Delta), \varepsilon) &= \sup_{z \in \mathbb{R}} \mathbf{P}\left(\left|X_\kappa(\Delta) - \left(z + \frac{\varepsilon}{2}\right)\right| \leq \frac{\varepsilon}{2}\right) \\ &\leq \sup_{z \in \mathbb{R}} \left( \frac{c_\kappa \left(\frac{\varepsilon}{2}\right)^{1/\kappa}}{\left(C\|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}^2 + \left(z + \frac{\varepsilon}{2}\right)^2\right)^{\frac{1}{2\kappa}}} \right) \leq \frac{c_\kappa \varepsilon^{1/\kappa}}{\left(2C\|\hat{K}_\Delta\|_{L_2(\mathbb{R}^{d\kappa})}\right)^{\frac{1}{\kappa}}} = C\varepsilon^{1/\kappa}. \end{aligned}$$

□

**REMARK 8.** Notice, that by Definitions 7 and 8

$$\begin{aligned} Q(X_\kappa(\Delta), \varepsilon) &= \sup_{z \in \mathbb{R}} (\mathbf{P}(X_\kappa(\Delta) \leq z + \varepsilon) - \mathbf{P}(X_\kappa(\Delta) \leq z)) \\ &= \sup_{z \in \mathbb{R}} |\mathbf{P}(X_\kappa(\Delta) \leq z) - \mathbf{P}(X_\kappa(\Delta) + \varepsilon \leq z)| = \rho(X_\kappa(\Delta) + \varepsilon, X_\kappa(\Delta)). \end{aligned}$$

**5. Rate of convergence.** In this section we consider the case of Hermite-type limit distributions in Theorem 2. The main result describes the rate of convergence of  $K_r$  to  $X_\kappa(\Delta)$  when  $r \rightarrow \infty$ . To prove it we use some techniques and facts from [5, 20, 18].

THEOREM 5. *Let Assumptions 1 and 2 hold and  $H\text{rank } G = \kappa \in \mathbb{N}$ .*

*If  $\tau \in (-\frac{d-\kappa\alpha}{2}, 0)$  then for any  $\varkappa < \frac{a}{2+a} \min\left(\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha}, \varkappa_1\right)$*

$$\rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) = o(r^{-\varkappa}), \quad r \rightarrow \infty,$$

*where  $a$  is a constant from Theorem 4,  $C_\kappa$  is defined by (2.2), and*

$$\varkappa_1 := \min\left(-2\tau, \frac{1}{\frac{1}{d-2\alpha} + \dots + \frac{1}{d-\kappa\alpha} + \frac{1}{d+1-\kappa\alpha}}\right).$$

*If  $\tau = 0$  then*

$$\rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) = g^{\frac{2}{3}}(r), \quad r \rightarrow \infty.$$

REMARK 9. This theorem generalises the result for the Rosenblatt-type case ( $\kappa = 2$ ) in [1] to Hermite-type asymptotics ( $\kappa > 2$ ). It also relaxes the assumptions on the spectral density used in [1], see Remarks 4 - 6.

PROOF. Since  $H\text{rank } G = \kappa$ , it follows that  $K_r$  can be represented in the space of squared-integrable random variables  $L_2(\Omega)$  as

$$K_r = K_{r,\kappa} + S_r := \frac{C_\kappa}{\kappa!} \int_{\Delta(r)} H_\kappa(\eta(x)) dx + \sum_{j \geq \kappa+1} \frac{C_j}{j!} \int_{\Delta(r)} H_j(\eta(x)) dx,$$

where  $C_j$  are coefficients of the Hermite series (2.2) of the function  $G(\cdot)$ .

Notice that  $\mathbf{E}K_{r,\kappa} = \mathbf{E}S_r = \mathbf{E}X_\kappa(\Delta) = 0$ , and

$$X_{r,\kappa} = \frac{\kappa! K_{r,\kappa}}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}.$$

It follows from Assumption 1 that  $|L(u)/u^\alpha| = |B(u)| \leq B(0) = 1$ . Thus, by the proof of Theorem 4 [20],

$$\begin{aligned} \mathbf{Var} S_r &\leq |\Delta|^2 r^{2d-(\kappa+1)\alpha} \sum_{j \geq \kappa+1} \frac{C_j^2}{j!} \int_0^{\text{diam}\{\Delta\}} z^{-(\kappa+1)\alpha} L^{\kappa+1}(rz) \psi_\Delta(z) dz \\ &\leq |\Delta|^2 r^{2d-\kappa\alpha} L^\kappa(r) \sum_{j \geq \kappa+1} \frac{C_j^2}{j!} \int_0^{\text{diam}\{\Delta\}} z^{-\kappa\alpha} \frac{L^\kappa(rz)}{L^\kappa(r)} \frac{L(rz)}{(rz)^\alpha} \psi_\Delta(z) dz. \end{aligned}$$



We represent the integral in (5) as the sum of two integrals  $I_1$  and  $I_2$  with the ranges of integration  $[0, r^{-\beta_1}]$  and  $(r^{-\beta_1}, \text{diam}\{\Delta\})$  respectively, where  $\beta_1 \in (0, 1)$ .

It follows from Assumption 1 that  $|L(u)/u^\alpha| = |B(u)| \leq B(0) = 1$  and we can estimate the first integral as

$$I_1 \leq \int_0^{r^{-\beta_1}} z^{-\kappa\alpha} \frac{L^\kappa(rz)}{L^\kappa(r)} \psi_\Delta(z) dz \leq \left( \frac{\sup_{0 \leq s \leq r} s^{\delta/\kappa} L(s)}{r^{\delta/\kappa} L(r)} \right)^\kappa \times \int_0^{r^{-\beta_1}} z^{-\delta} z^{-\kappa\alpha} \psi_\Delta(z) dz,$$

where  $\delta$  is an arbitrary number in  $(0, \min(\alpha, d - \kappa\alpha))$ .

By Assumption 1 the function  $L(\cdot)$  is locally bounded. By Theorem 1.5.3 in [4], there exists  $r_0 > 0$  and  $C > 0$  such that for all  $r \geq r_0$

$$\frac{\sup_{0 \leq s \leq r} s^{\delta/2} L(s)}{r^{\delta/2} L(r)} \leq C.$$

Using (2.1) we obtain

$$\int_0^{r^{-\beta_1}} z^{-\delta} z^{-\kappa\alpha} \psi_\Delta(z) dz \leq \frac{C}{|\Delta|} \int_0^{r^{-\beta_1}} \tau^{d-\kappa\alpha-1-\delta} d\tau = \frac{C r^{-\beta_1(d-\kappa\alpha-\delta)}}{(d-\kappa\alpha-\delta)|\Delta|}.$$

Applying Theorem 1.5.3 [4] we get

$$I_2 \leq \frac{\sup_{r^{1-\beta_1} \leq s \leq r \cdot \text{diam}\{\Delta\}} s^\delta L^\kappa(s)}{r^\delta L^\kappa(r)} \cdot \sup_{r^{1-\beta_1} \leq s \leq r \cdot \text{diam}\{\Delta\}} \frac{L(s)}{s^\alpha} \int_0^{\text{diam}\{\Delta\}} z^{-(\delta+\kappa\alpha)} \psi_\Delta(z) dz \leq C \cdot o(r^{-(\alpha-\delta)(1-\beta_1)}),$$

when  $r$  is sufficiently large.

Notice that by (2.3)

$$\sum_{j \geq \kappa+1} \frac{C_j^2}{j!} \leq \int_{\mathbb{R}} G^2(w) \phi(w) dw < +\infty.$$

Hence, for sufficiently large  $r$

$$\mathbf{Var} S_r \leq C r^{2d-\kappa\alpha} L^\kappa(r) \left( r^{-\beta_1(d-\kappa\alpha-\delta)} + o\left(r^{-(\alpha-\delta)(1-\beta_1)}\right) \right).$$

Choosing  $\beta_1 = \frac{\alpha}{d-(\kappa-1)\alpha}$  to minimize the upper bound we get

$$\mathbf{Var} S_r \leq C r^{2d-\kappa\alpha} L^\kappa(r) r^{-\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha} + \delta}.$$

It follows from Theorem 4 that

$$\rho(X_\kappa(\Delta) + \varepsilon, X_\kappa(\Delta)) \leq C\varepsilon^a.$$

Applying Chebyshev's inequality and Lemma 3 to  $X = X_{r,\kappa}$ ,  $Y = \frac{\kappa! S_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}$ , and  $Z = X_\kappa(\Delta)$ , we get

$$\begin{aligned} \rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) &= \rho\left(X_{r,\kappa} + \frac{\kappa! S_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) \\ &\leq \rho(X_{r,\kappa}, X_\kappa(\Delta)) + C\left(\varepsilon^a + \varepsilon^{-2} r^{-\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha} + \delta}\right), \end{aligned}$$

for a sufficiently large  $r$ .

Choosing  $\varepsilon := r^{-\frac{\alpha(d-\kappa\alpha)}{(2+a)(d-(\kappa-1)\alpha)}}$  to minimize the second term we obtain

$$(5.1) \quad \rho\left(\frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta)\right) \leq \rho(X_{r,\kappa}, X_\kappa(\Delta)) + C r^{\frac{-a\alpha(d-\kappa\alpha)}{(2+a)(d-(\kappa-1)\alpha)} + \delta}.$$

Applying Lemma 3 once again to  $X = X_\kappa(\Delta)$ ,  $Y = X_{r,\kappa} - X_\kappa(\Delta)$ , and  $Z = X_\kappa(\Delta)$  we obtain

$$(5.2) \quad \begin{aligned} \rho(X_{r,\kappa}, X_\kappa(\Delta)) &\leq \varepsilon_1^a C + P\{|X_{r,\kappa} - X_\kappa(\Delta)| \geq \varepsilon_1\} \\ &\leq \varepsilon_1^a C + \varepsilon_1^{-2} \mathbf{Var}(X_{r,\kappa} - X_\kappa(\Delta)). \end{aligned}$$

Now we show how to estimate  $\mathbf{Var}(X_{r,\kappa} - X_\kappa(\Delta))$ .

By the self-similarity of Gaussian white noise and formula (2.1) [10]

$$\begin{aligned} X_{r,\kappa} &\stackrel{\mathcal{D}}{=} c_2^{\frac{\kappa}{2}}(d, \alpha) \int'_{\mathbb{R}^{\kappa d}} K_\Delta(\lambda_1 + \dots + \lambda_\kappa) Q_r(\lambda_1, \dots, \lambda_\kappa) \\ &\quad \times \frac{W(d\lambda_1) \dots W(d\lambda_\kappa)}{\|\lambda_1\|^{(d-\kappa\alpha)/2} \dots \|\lambda_\kappa\|^{(d-\kappa\alpha)/2}}, \end{aligned}$$

where

$$Q_r(\lambda_1, \dots, \lambda_\kappa) := r^{\frac{\kappa}{2}(\alpha-d)} L^{-\frac{\kappa}{2}}(r) c_2^{-\frac{\kappa}{2}}(d, \alpha) \left[ \prod_{i=1}^{\kappa} \|\lambda_i\|^{d-\alpha} f\left(\frac{\|\lambda_i\|}{r}\right) \right]^{1/2}.$$

Notice that

$$X_\kappa(\Delta) = c_2^{\frac{\kappa}{2}}(d, \alpha) \int'_{\mathbb{R}^{\kappa d}} K_\Delta(\lambda_1 + \dots + \lambda_\kappa) \frac{W(d\lambda_1) \dots W(d\lambda_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \dots \|\lambda_\kappa\|^{(d-\alpha)/2}}.$$

By the isometry property of multiple stochastic integrals

$$\begin{aligned} R_r &:= \frac{\mathbb{E} |X_{r,\kappa} - X_\kappa(\Delta)|^2}{c_2^\kappa(d, \alpha)} \\ &= \int_{\mathbb{R}^{\kappa d}} \frac{|K_\Delta(\lambda_1 + \dots + \lambda_\kappa)|^2 (Q_r(\lambda_1, \dots, \lambda_\kappa) - 1)^2}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_\kappa\|^{d-\alpha}} d\lambda_1 \dots d\lambda_\kappa. \end{aligned}$$

Let us rewrite the integral  $R_r$  as the sum of two integrals  $I_3$  and  $I_4$  with the integration regions  $A(r) := \{(\lambda_1, \dots, \lambda_\kappa) \in \mathbb{R}^{\kappa d} : \max_{i=1, \dots, \kappa} (\|\lambda_i\|) \leq r^\gamma\}$  and  $\mathbb{R}^{\kappa d} \setminus A(r)$  respectively, where  $\gamma \in (0, 1)$ . Our intention is to use the monotone equivalence property of regularly varying functions in the regions  $A(r)$ .

First we consider the case of  $(\lambda_1, \dots, \lambda_\kappa) \in A(r)$ . By Assumption 2 and the inequality

$$\left| \sqrt{\prod_{i=1}^{\kappa} x_i} - 1 \right| \leq \sum_{i=1}^{\kappa} \left| x_i^{\frac{\kappa}{2}} - 1 \right|$$

we obtain

$$|Q_r(\lambda_1, \dots, \lambda_2) - 1| = \left| \sqrt{\prod_{j=1}^{\kappa} \frac{L\left(\frac{r}{\|\lambda_j\|}\right)}{L(r)}} - 1 \right| \leq \sum_{j=1}^{\kappa} \left| \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} - 1 \right|.$$

By Lemma 1, if  $\|\lambda_j\| \in (1, r^\gamma)$ ,  $j = \overline{1, \kappa}$ , then for arbitrary  $\delta_1 > 0$  and sufficiently large  $r$  we get

$$\begin{aligned} \left| 1 - \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} \right| &= \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} \cdot \left| 1 - \frac{L^{\frac{\kappa}{2}}(r)}{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)} \right| \leq C \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} g\left(\frac{r}{\|\lambda_j\|}\right) \\ &\times \|\lambda_j\|^{\delta_1} h_\tau(\|\lambda_j\|) = C \|\lambda_j\|^{\delta_1} h_\tau(\|\lambda_j\|) g(r) \frac{g\left(\frac{r}{\|\lambda_j\|}\right)}{g(r)} \left( \frac{L\left(\frac{r}{\|\lambda_j\|}\right)}{L(r)} \right)^{\frac{\kappa}{2}}. \end{aligned}$$

For any positive  $\beta_2$  and  $\beta_3$ , applying Theorem 1.5.6 [4] to  $g(\cdot)$  and  $L(\cdot)$  and using the fact that  $h_\tau\left(\frac{1}{t}\right) = -\frac{1}{t^\tau}h(t)$  we obtain

$$\begin{aligned} \left| 1 - \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} \right| &\leq C \|\lambda_j\|^{\delta_1 + \frac{\kappa\beta_2}{2} + \beta_3} \|\lambda_j\|^{-\tau} h_\tau(\|\lambda_j\|) g(r) \\ (5.3) \quad &= C \|\lambda_j\|^\delta h_\tau\left(\frac{1}{\|\lambda_j\|}\right) g(r). \end{aligned}$$

By Lemma 1 for  $\|\lambda_j\| \leq 1$ ,  $j = \overline{1, \kappa}$ , and arbitrary  $\delta > 0$ , we obtain

$$(5.4) \quad \left| 1 - \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} \right| \leq C \|\lambda_j\|^{-\delta} h_\tau\left(\frac{1}{\|\lambda_j\|}\right) g(r).$$

Hence, by (5.3) and (5.4)

$$\begin{aligned} |Q_r(\lambda_1, \dots, \lambda_\kappa) - 1|^2 &\leq k \sum_{j=1}^{\kappa} \left| \frac{L^{\frac{\kappa}{2}}\left(\frac{r}{\|\lambda_j\|}\right)}{L^{\frac{\kappa}{2}}(r)} - 1 \right|^2 \\ &\leq C \sum_{j=1}^{\kappa} h_\tau^2\left(\frac{1}{\|\lambda_j\|}\right) g^2(r) \max\left(\|\lambda_j\|^{-\delta}, \|\lambda_j\|^\delta\right), \end{aligned}$$

for  $(\lambda_1, \dots, \lambda_\kappa) \in A(r)$ .

Notice, that in the case  $\tau = 0$  for any  $\delta > 0$  there exists  $C > 0$  such that  $h_0(x) = \ln(x) < Cx^\delta$ ,  $x \geq 1$ , and  $h_0(x) = \ln(x) < Cx^{-\delta}$ ,  $x < 1$ . Hence, by Lemma 2 for  $-\tau \leq \frac{d-\kappa\alpha}{2}$  we get

$$\int_{A(r) \cap [0,1]^{\kappa d}} \frac{h_\tau^2\left(\frac{1}{\|\lambda_j\|}\right) \max\left(\|\lambda_j\|^{-\delta}, \|\lambda_j\|^\delta\right) \left| K_\Delta\left(\sum_{i=1}^{\kappa} \lambda_i\right) \right|^2 d\lambda_1 \dots d\lambda_\kappa}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_\kappa\|^{d-\alpha}} < \infty.$$

Therefore, we obtain for sufficiently large  $r$

$$I_3 \leq C g^2(r) \sum_{j=1}^{\kappa} \int_{A(r) \cap \mathbb{R}^{\kappa d}} \frac{h_\tau^2\left(\frac{1}{\|\lambda_j\|}\right) \cdot \max\left(\|\lambda_j\|^{-\delta}, \|\lambda_j\|^\delta\right)}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_\kappa\|^{d-\alpha}}$$

$$\begin{aligned}
& \times |K_{\Delta}(\lambda_1 + \dots \lambda_{\kappa})|^2 d\lambda_1 \dots d\lambda_{\kappa} \leq C g^2(r) \int_{A(r) \cap \mathbb{R}^{\kappa d}} \frac{h_{\tau}^2\left(\frac{1}{\|\lambda_1\|}\right)}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \\
(5.5) \quad & \times \max\left(\|\lambda_1\|^{-\delta}, \|\lambda_1\|^{\delta}\right) |K_{\Delta}(\lambda_1 + \dots \lambda_{\kappa})|^2 d\lambda_1 \dots d\lambda_{\kappa} \leq C g^2(r).
\end{aligned}$$

It follows from Assumption 2 and the specification of the estimate (23) in the proof of Theorem 5 [20] that for each positive  $\delta$  there exists  $r_0 > 0$  such that for all  $r \geq r_0$ ,  $(\lambda_1, \dots, \lambda_{\kappa}) \in B_{(1, \mu_2, \dots, \mu_{\kappa})} = \{(\lambda_1, \dots, \lambda_{\kappa}) \in \mathbb{R}^{\kappa d} : \|\lambda_j\| \leq 1, \text{ if } \mu_j = -1, \text{ and } \|\lambda_j\| > 1, \text{ if } \mu_j = 1, j = \overline{1, k}\}, \text{ and } \mu_j \in \{-1, 1\}, \text{ it holds}$

$$\begin{aligned}
& \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 (Q_r(\lambda_1, \dots, \lambda_{\kappa}) - 1)^2}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \leq \frac{C |K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \\
& + C \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2}{\|\lambda_1\|^{d-\alpha-\delta} \|\lambda_2\|^{d-\alpha-\mu_2\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}}.
\end{aligned}$$

Since the integrands are non-negative, we can estimate  $I_4$  as it is shown below

$$\begin{aligned}
I_4 & \leq \kappa \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 (Q_r(\lambda_1, \dots, \lambda_{\kappa}) - 1)^2 d\lambda_1 \dots d\lambda_{\kappa}}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \\
& \leq C \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_2)|^2 d\lambda_1 \dots d\lambda_{\kappa}}{\|\lambda_1\|^{d-\alpha} \dots \|\lambda_{\kappa}\|^{d-\alpha}} \\
& + C \sum_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in \overline{1, \kappa}}} \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} \frac{|K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 d\lambda_1 \dots d\lambda_{\kappa}}{\|\lambda_1\|^{d-\alpha-\delta} \|\lambda_2\|^{d-\alpha-\mu_2\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}} \\
& \leq C \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in \overline{2, \kappa}}} \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} |K_{\Delta}(\lambda_1 + \dots + \lambda_{\kappa})|^2 \\
(5.6) \quad & \times \frac{d\lambda_1 \dots d\lambda_{\kappa}}{\|\lambda_1\|^{d-\alpha-\delta} \|\lambda_2\|^{d-\alpha-\mu_2\delta} \dots \|\lambda_{\kappa}\|^{d-\alpha-\mu_{\kappa}\delta}}.
\end{aligned}$$

Replacing  $\lambda_1 + \lambda_2$  by  $u$  we obtain

$$I_4 \leq C \max_{\substack{\mu_i \in \{0, 1, -1\} \\ i \in \overline{2, \kappa}}} \int_{\mathbb{R}^{(\kappa-1)d}} \int_{\|\lambda_1\| > r^{\gamma}} \frac{|K_{\Delta}(u + \lambda_3 + \dots + \lambda_{\kappa})|^2}{\|\lambda_1\|^{d-\alpha-\delta} \|u - \lambda_1\|^{d-\alpha-\mu_2\delta}}$$

$$\begin{aligned} & \times \frac{d\lambda_1 du d\lambda_3 \dots d\lambda_\kappa}{\|\lambda_3\|^{d-\alpha-\mu_3\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \leq C \max_{\substack{\mu_i \in \{0,1,-1\} \\ i \in \{2,\kappa\}}} \int_{\mathbb{R}^{(\kappa-1)d}} \frac{1}{\|u\|^{d-2\alpha-(\mu_2+1)\delta}} \\ & \times \frac{|K_\Delta(u + \lambda_3 + \dots + \lambda_\kappa)|^2}{\|\lambda_3\|^{d-\alpha-\mu_3\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \int_{\|\lambda_1\| > \frac{r_0^\gamma}{\|u\|}} \frac{d\lambda_1 du d\lambda_3 \dots d\lambda_\kappa}{\|\lambda_1\|^{d-\alpha-\delta} \left\| \frac{u}{\|u\|} - \lambda_1 \right\|^{d-\alpha-\mu_2\delta}}. \end{aligned}$$

Taking into account that for  $\delta \in (0, \min(\alpha, d/\kappa - \alpha))$

$$\sup_{u \in \mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} \frac{d\lambda_1}{\|\lambda_1\|^{d-\alpha-\delta} \left\| \frac{u}{\|u\|} - \lambda_1 \right\|^{d-\alpha-\mu_2\delta}} \leq C,$$

we obtain

$$\begin{aligned} I_4 & \leq C \max_{\substack{\mu_i \in \{0,1,-1\} \\ i \in \{3,\kappa\}}} \int_{\mathbb{R}^{(\kappa-2)d}} \left[ \max_{\substack{\mu_2 \in \{0,1,-1\} \\ \|u\| \leq r_0^\gamma}} \int \frac{|K_\Delta(u + \lambda_3 + \dots + \lambda_\kappa)|^2}{\|u\|^{d-2\alpha-(\mu_2+1)\delta}} \right. \\ & \times \frac{d\lambda_3 \dots d\lambda_\kappa}{\|\lambda_3\|^{d-\alpha-\mu_3\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \int_{\|\lambda_1\| > r^{\gamma-\gamma_0}} \frac{d\lambda_1 du}{\|\lambda_1\|^{d-\alpha-\delta} \left\| \frac{u}{\|u\|} - \lambda_1 \right\|^{d-\alpha-\mu_2\delta}} \\ & \left. + \max_{\substack{\mu_i \in \{0,1,-1\} \\ \|u\| > r_0^\gamma}} \int \frac{|K_\Delta(u + \lambda_3 + \dots + \lambda_\kappa)|^2 du d\lambda_3 \dots d\lambda_\kappa}{\|u\|^{d-2\alpha-(\mu_2+1)\delta} \|\lambda_3\|^{d-\alpha-\mu_3\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \right], \end{aligned}$$

where  $\gamma_0 \in (0, \gamma)$ .

By Lemma 2, there exists  $r_0 > 0$  such that for all  $r \geq r_0$  the first summand is bounded by

$$\begin{aligned} & C \max_{\substack{\mu_i \in \{0,1,-1\} \\ \|u\| \leq r_0^\gamma}} \int \frac{|K_\Delta(u + \lambda_3 + \dots + \lambda_\kappa)|^2 du d\lambda_3 \dots d\lambda_\kappa}{\|u\|^{d-2\alpha-(\mu_2+1)\delta} \|\lambda_3\|^{d-\alpha-\mu_3\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}} \\ & \times \int_{\|\lambda_1\| > r^{\gamma-\gamma_0}} \frac{d\lambda_1}{\|\lambda_1\|^{2d-2\alpha-\delta-\mu_2\delta}} \leq C r^{-(\gamma-\gamma_0)(d-2\alpha-2\delta)}. \end{aligned}$$

Therefore, for sufficiently large  $r$ ,

$$\begin{aligned} I_4 & \leq C r^{-(\gamma-\gamma_0)(d-2\alpha-2\delta)} \\ & + C \max_{\substack{\mu_i \in \{0,1,-1\} \\ i \in \{3,\kappa\}}} \int_{\mathbb{R}^{(\kappa-2)d}} \int_{\|u\| > r^{\gamma_0}} \frac{|K_\Delta(u + \lambda_3 + \dots + \lambda_\kappa)|^2 du d\lambda_3 \dots d\lambda_\kappa}{\|u\|^{d-2\alpha-2\delta} \|\lambda_3\|^{d-\alpha-\mu_3\delta} \dots \|\lambda_\kappa\|^{d-\alpha-\mu_\kappa\delta}}. \end{aligned}$$

Notice that the second summand here coincides with (5.6) if  $\kappa$  is replaced by  $\kappa - 1$ . Thus, we can repeat the above procedure  $\kappa - 2$  more times and get the result

$$(5.7) \quad I_4 \leq C r^{-(\gamma-\gamma_0)(d-2\alpha-2\delta)} + \dots + C r^{-(\gamma_{\kappa-3}-\gamma_{\kappa-2})(d-\kappa\alpha-\kappa\delta)} \\ + C \int_{\|u\| > r^{\gamma_{\kappa-2}}} \frac{|K_{\Delta}(u)|^2 du}{\|u\|^{d-\kappa\alpha-\kappa\delta}},$$

where  $\gamma > \gamma_0 > \gamma_1 > \dots > \gamma_{\kappa-2}$ .

By the spherical  $L_2$ -average decay rate of the Fourier transform [5] for  $\delta < d + 1 - \kappa\alpha$  and sufficiently large  $r$  we get the following estimate of the integral in (5.7)

$$(5.8) \quad \int_{\|u\| > r^{\gamma_{\kappa-2}}} \frac{|K_{\Delta}(u)|^2 du}{\|u\|^{d-\kappa\alpha-\kappa\delta}} \leq C \int_{z > r^{\gamma_{\kappa-2}}} \int_{S^{d-1}} \frac{|K_{\Delta}(z\omega)|^2}{z^{1-\kappa\alpha-\kappa\delta}} d\omega dz \\ \leq C \int_{z > r^{\gamma_{\kappa-2}}} \frac{dz}{z^{d+2-\kappa\alpha-\kappa\delta}} = C r^{-\gamma_{\kappa-2}(d+1-\kappa\alpha-\kappa\delta)} \\ = C r^{-(\gamma_{\kappa-2}-\gamma_{\kappa-1})(d+1-\kappa\alpha-\kappa\delta)},$$

where  $S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$  is a sphere of radius 1 in  $\mathbb{R}^d$  and  $\gamma_{\kappa-1} = 0$ .

Now let's consider the case  $\tau < 0$ . In this case by Theorem 1.5.6 [4] for any  $\delta > 0$  we can estimate  $g(r)$  as follows

$$(5.9) \quad g(r) \leq C r^{\tau+\delta}.$$

Combining estimates (5.1), (5.2), (5.5), (5.7), (5.8), (5.9) and choosing  $\varepsilon_1 := r^{-\beta}$ , we obtain

$$\rho \left( \frac{\kappa! K_r}{C_{\kappa} r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_{\kappa}(\Delta) \right) \leq C \left( r^{-\frac{a\alpha(d-\kappa\alpha)}{(2+a)(d-(\kappa-1)\alpha)}+\delta} + r^{-a\beta} + r^{2\tau+2\delta+2\beta} \right. \\ \left. + r^{-(\gamma-\gamma_0)(d-2\alpha-2\delta)+2\beta} + \dots + r^{-(\gamma_{\kappa-3}-\gamma_{\kappa-2})(d-\kappa\alpha-\kappa\delta)+2\beta} \right. \\ \left. + r^{-(\gamma_{\kappa-2}-\gamma_{\kappa-1})(d+1-\kappa\alpha-\kappa\delta)+2\beta} \right).$$

Therefore, for any  $\tilde{\varkappa}_1 \in (0, \frac{2+a}{a}\varkappa_0)$  one can choose a sufficiently small  $\delta > 0$  such that

$$(5.10) \quad \rho \left( \frac{\kappa! K_r}{C_{\kappa} r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_{\kappa}(\Delta) \right) \leq C r^{\delta} \left( r^{-\frac{a\alpha(d-\kappa\alpha)}{(2+a)(d-(\kappa-1)\alpha)}} + r^{-\frac{a\tilde{\varkappa}_1}{2+a}} \right),$$

where

$$\varkappa_0 := \sup_{\substack{1 > \gamma > \gamma_0 > \dots > \gamma_{\kappa-1} = 0 \\ \beta > 0}} \min(a\beta, -2\tau - 2\beta, (\gamma - \gamma_0)(d - 2\alpha) - 2\beta, \dots,$$

$$(\gamma_{\kappa-3} - \gamma_{\kappa-2})(d - \kappa\alpha) - 2\beta, (\gamma_{\kappa-2} - \gamma_{\kappa-1})(d + 1 - \kappa\alpha) - 2\beta).$$

LEMMA 5. Let  $\mathbf{\Gamma} = \{\gamma = (\gamma_1, \dots, \gamma_{n+1}) \mid b = \gamma_0 > \gamma_1 > \dots > \gamma_{n+1} = 0\}$  and  $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}_+^{n+1}$  be some fixed vector.

The function  $G(\gamma) = \min_i (\gamma_i - \gamma_{i+1}) x_i$  reaches its maximum at  $\bar{\gamma} = (\bar{\gamma}_0, \dots, \bar{\gamma}_{n+1}) \in \mathbf{\Gamma}$  such that for any  $0 \leq i \leq n$  it holds

$$(5.11) \quad (\bar{\gamma}_i - \bar{\gamma}_{i+1}) x_i = (\bar{\gamma}_{i+1} - \bar{\gamma}_{i+2}) x_{i+1}.$$

PROOF. Let us show that any deviation of  $\gamma$  from  $\bar{\gamma}$  leads to a smaller result. Consider a vector  $\hat{\gamma}$  such that for some  $i \in \overline{1, n}$  and some  $\varepsilon > 0$  the following relation is true

$$\hat{\gamma}_i - \hat{\gamma}_{i+1} = \bar{\gamma}_i - \bar{\gamma}_{i+1} + \varepsilon.$$

Since  $\sum_{i=0}^n \hat{\gamma}_i - \hat{\gamma}_{i+1} = \hat{\gamma}_0 - \hat{\gamma}_{n+1} = b$  we can conclude that there exist some  $j \neq i$ ,  $j \in \overline{1, n}$ , and  $\varepsilon_1 > 0$  such that  $\hat{\gamma}_j - \hat{\gamma}_{j+1} = \bar{\gamma}_j - \bar{\gamma}_{j+1} - \varepsilon_1$ .

Obviously, in this case

$$G(\hat{\gamma}) \leq (\hat{\gamma}_j - \hat{\gamma}_{j+1}) x_j = (\bar{\gamma}_j - \bar{\gamma}_{j+1} - \varepsilon_1) x_j = (\bar{\gamma}_j - \bar{\gamma}_{j+1}) x_j - \varepsilon_1 x_j$$

Since  $\varepsilon_1 > 0$  and  $x_j > 0$  it follows from (5.11) that

$$G(\hat{\gamma}) \leq (\bar{\gamma}_j - \bar{\gamma}_{j+1}) x_j - \varepsilon_1 x_j < (\bar{\gamma}_j - \bar{\gamma}_{j+1}) x_j = G(\bar{\gamma}).$$

So it's clearly seen that any deviation from  $\bar{\gamma}$  will yield a smaller result.  $\square$

Note, that for fixed  $\gamma \in (0, 1)$  by Lemma 5

$$\sup_{\gamma > \gamma_0 > \dots > \gamma_{\kappa-1} = 0} \min((\gamma - \gamma_0)(d - 2\alpha), \dots, (\gamma_{\kappa-3} - \gamma_{\kappa-2})(d - \kappa\alpha),$$

$$(\gamma_{\kappa-2} - \gamma_{\kappa-1})(d + 1 - \kappa\alpha)) = \frac{\gamma}{\frac{1}{d-2\alpha} + \dots + \frac{1}{d-\kappa\alpha} + \frac{1}{d+1-\kappa\alpha}}$$

and

$$\sup_{\gamma \in (0, 1)} \frac{\gamma}{\frac{1}{d-2\alpha} + \dots + \frac{1}{d-\kappa\alpha} + \frac{1}{d+1-\kappa\alpha}} = \frac{1}{\frac{1}{d-2\alpha} + \dots + \frac{1}{d-\kappa\alpha} + \frac{1}{d+1-\kappa\alpha}}.$$



Note that  $\varkappa_0 = \sup_{\beta > 0} \min(a\beta, \varkappa_1 - 2\beta) = \frac{a\varkappa_1}{2+a}$ .

Finally, from (5.10) for  $\tilde{\varkappa}_1 < \varkappa_1$  the first statement of the theorem follows.

Now let's consider the case  $\tau = 0$ . In this case by Theorem 1.5.6 [4] for any  $s > 0$  and sufficiently large  $r$

$$(5.12) \quad g(r) > r^{-s}.$$

Combining estimates (5.1), (5.2), (5.5), (5.7), (5.8), replacing all powers of  $r$  for  $g^2(r)$  using (5.12), and choosing  $\varepsilon_1 := g^\beta(r)$ ,  $\beta \in (0, 1)$  we obtain

$$\rho \left( \frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta) \right) \leq C \left( g^2(r) + g^\beta(r) + g^{2-2\beta}(r) \right).$$

Since  $\sup_{\beta \in (0,1)} \min(2, \beta, 2-2\beta) = \frac{2}{3}$ , it follows that

$$\rho \left( \frac{\kappa! K_r}{C_\kappa r^{d-\frac{\kappa\alpha}{2}} L^{\frac{\kappa}{2}}(r)}, X_\kappa(\Delta) \right) \leq C g^{\frac{2}{3}}(r).$$

This proves the second statement of the theorem.  $\square$

**REMARK 10.** The upper bound on the rate of convergence in Theorem 5 is given by explicit formulae that are easy to evaluate and analyse. For example, for fixed values of  $\alpha$  and  $\kappa$  it is simple to see that the upper bound for  $\varkappa$  approaches  $\frac{a}{2+a} \min(\alpha, -2\tau)$ , when  $d \rightarrow +\infty$ . For fixed values of  $d$  and  $\kappa$  the upper bound for  $\varkappa$  is of the order of magnitude of  $O(d - \kappa\alpha)$ , when  $\alpha \rightarrow d/\kappa$ . This result is expected as the value  $\alpha = d/\kappa$  corresponds to the boundary where a phase transition between short- and long-range dependence occurs.

**6. Conclusion.** The rate of convergence to Hermite-type limit distributions in non-central limit theorems was investigated. The results were obtained under rather general assumptions on the spectral densities of the considered random fields, that weaken the assumptions used in [1]. Similar to [1], the direct probabilistic approach was used, which has, in our view, an independent interest as an alternative to the methods in [6, 25, 26]. Additionally, some fine properties of the probability distributions of Hermite-type random variables were investigated. Some special cases when their probability density functions are bounded were discussed. New anti-concentration inequalities were derived for Lévy concentration functions.

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